Integrable su(3) spin chain combining different representations

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Abstract. The general expression for the local matrix $t(\theta)$ of a quantum chain with the site space in any representation of su(3) is obtained. This is made by generalizing $t(\theta)$ from the fundamental representation and imposing the fulfilment of the Yang–Baxter equation. Then, a non-homogeneous spin chain combining different representations of su(3) is solved by developing a method inspired in the nested Bethe ansatz. The solution for the eigenvalues of the trace of the monodromy matrix is given as two coupled Bethe equations. A conjecture about the solution of a chain with the site states in different representations of su(n) is presented. The thermodynamic limit of the ground state is calculated.

1. Introduction

The search for integrable spin chains has gained considerable attention recently due to the fact that they are interesting physical systems and have a rich mathematical structure. The best known is the XXZ Heisenberg su(2) chain with spin $S = \frac{1}{2}$ in every site [1], which gave rise to the subsequent development of the quantum groups [2–4]. Integrable spin chains with $S = 1$ and higher spin chains have been found and solved [5–11]. They correspond to higher dimension representations of the quantum group that give integrable systems of increasing complexity [12–14].

In addition, magnetic Hamiltonians can be derived from the solution of the Yang–Baxter equations (YBE) [15, 16] associated with Lie algebras other than su(2) [17]. The solutions are found using the Bethe ansatz (BA) for sites with two components or nested Bethe ansatz (NBA) for sites with more components [18]. The introduction of the quantum inverse scattering methods (QISM) [19] gave a systematic method to solve those systems. The quantum groups give general methods to find new integrable models.

An interesting problem is to solve integrable chains formed by two kinds of states of the site. Inhomogeneus solvable models were considered in [20] (see also [12]). The simplest case, an alternating chain with $S = \frac{1}{2}$ and $S = 1$ derived from the su(2) Lie algebra was presented in [21] and in several subsequent works in which the thermodynamic properties of these systems were studied [22–25].

The system presents interesting features; one of them is that it gives a Hamiltonian that contains the usual piece coupling pairs of neighbouring spins $S = \frac{1}{2}$ and $S = 1$ and another piece coupling three neighbouring spins. The solution is found using the BA.

In this paper, we are going to solve an alternating chain with the spin of the sites in the {3} and {3*} representations of su(3). We made an extension of the method used in [21] for

systems where the P and T symmetries are not conserved in order to obtain Hamiltonians associated to alternating chains based on the su(2) algebra.

In a more rigorous sense, we are using the $U_q(\text{su}(3))$ algebra and its representations, but it can be shown that generally for simple algebras g the representations of g and $U_q(g)$ are isomorphic [26].

We can obtain two different systems by using as auxiliary spaces the representations {3} and {3*}; they will give different Hamiltonians, but under a relation between the parameters of the local inhomogeneities that we will specified, we can prove that they commute and both systems have the same eigenstates. Then, the more general system will be a superposition of those two systems.

The diagonalization of these Hamiltonians requires important modifications of the standard method with the NBA [17, 20]. We start by building the monodromy matrix in the auxiliary space whose elements are operators in the space of states of the chain. The main difference is that, in these new systems, we do not have a common eigenstate of the operators in the diagonal of the monodromy matrix which is annihilated by all operators below the diagonal of this matrix. Then, we introduce a reference subspace in the space of states where we can do the second step of the NBA. So, we obtain the equations of the ansatz whose properties can be analysed as in the standard case. The model, since the auxiliary space has three dimensions, requires only two steps for the NBA, but the method is easily generalizable to more dimensions [27].

This paper is organized as follows. In section 2 we develop a technique to obtain the Hamiltonians associated to alternating chains [21]. In section 3 we apply the method to alternating chains with the sites in the {3} and {3*} representations of su(3). In section 4 we find the eigenvalues of transfer matrix of the system and the equations of our ansatz as a generalization of the NBA. In section 5, we analyse the equations of the ansatz and obtain their thermodynamic limit.

2. Non-homogeneous chain with the site states alternating in two different representation spaces

As is well known, regular solutions of the YBE systematically yield integrable chains. In [21] an integrable quantum chain with two types of spins is described. Following [21] and in order to establish our notation, we are going to review how an integrable system follows from a R-matrix $R_{c,a}^{b,d}(\theta)$, which is a solution of the YBE

$$[1 \otimes R(\theta - \theta')][R(\theta) \otimes 1][1 \otimes R(\theta')] = [R(\theta') \otimes 1][1 \otimes R(\theta)][R(\theta - \theta') \otimes 1]. \qquad (2.1)$$
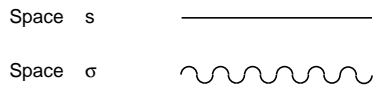
We associate to each site of the chain the t operator

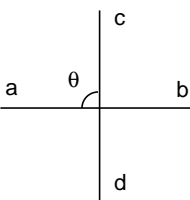$$[t_{a,b}(\theta)]_{c,d} = R_{c,a}^{b,d}(\theta) \qquad (2.2)$$

where the indices a and b act on the site space and the c and d in an auxiliary space. They are shown graphically in figure 1(a). Then the YBE can be written in the usual form,

$$R(\theta - \theta') \cdot [t(\theta) \otimes t(\theta')] = [t(\theta') \otimes t(\theta)] \cdot R(\theta - \theta') \qquad (2.3)$$
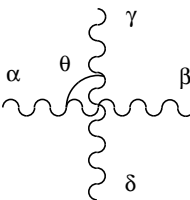
that is graphically expressed in figure 2(a). The $\otimes$ product is in the site space and the $\cdot$ product is in the auxiliary space.

Equation (2.1) is not the most general YBE. In general we have operators acting on pairs of unequal vector spaces. This is represented graphically with lines of different kinds.

Space s

Space σ

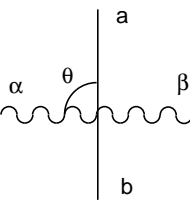$$[t_{a,b}(\theta)]_{c,d} = R^{b,d}_{c,a}(\theta) =$$
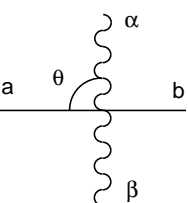
(*a*)

$$[t^{*}_{\alpha,\beta}(\theta)]_{\gamma,\delta} = R^{*\beta,\delta}_{\gamma,\alpha}(\theta) =$$

(*b*)

$$[\bar{t}_{a,b}(\theta)]_{\alpha,\beta} =$$

(*c*)

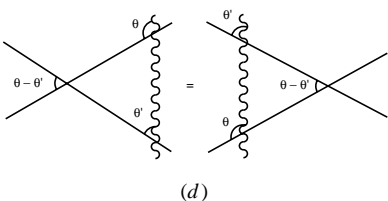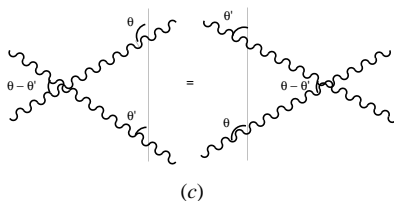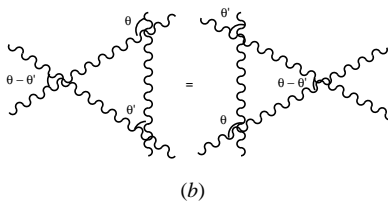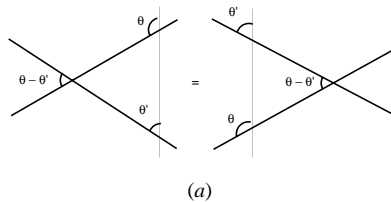$$[\tilde{t}_{\alpha,\beta}(\theta)]_{a,b} =$$

(*d*)

Figure 1.

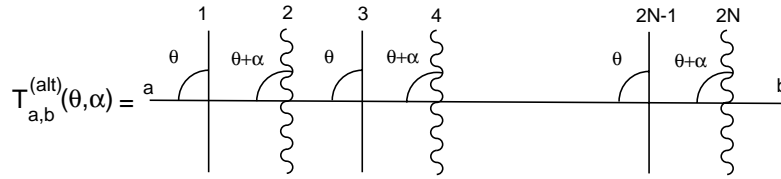(*a*)   (*b*)

(*c*)   (*d*)

Figure 2.

Figure 3.

We are going to consider two vector spaces denoted by s and σ; then we have, besides t, the operators $t^* = R^*$, $\bar{t}$ and $\tilde{t}$ represented in figure 1. They fulfil the YBEs,

$$R^*(\theta - \theta') \cdot [t^*(\theta) \otimes t^*(\theta')] = [t^*(\theta') \otimes t^*(\theta)] \cdot R^*(\theta - \theta') \tag{2.4a}$$

$$R^*(\theta - \theta') \cdot [\tilde{t}(\theta) \otimes \tilde{t}(\theta')] = [\tilde{t}(\theta') \otimes \tilde{t}(\theta)] \cdot R^*(\theta - \theta') \tag{2.4b}$$

$$R(\theta - \theta') \cdot [\bar{t}(\theta) \otimes \bar{t}(\theta')] = [\bar{t}(\theta') \otimes \bar{t}(\theta)] \cdot R(\theta - \theta') \tag{2.4c}$$

represented in figures 2(*b*)–(*d*) respectively.

In the most general case, we do not require $R(\theta)$ and $R^*(\theta)$ to have P and T symmetry nor to be invariant under crossing. Instead, we will assume the following properties.

(i) PT-symmetry,

$$R_{a,b}^{c,d}(\theta) = R_{c,d}^{a,b}(\theta) \tag{2.5a}$$

$$R^{*\gamma,\delta}_{\alpha,\beta}(\theta) = R^{*\alpha,\beta}_{\gamma,\delta}(\theta). \tag{2.5b}$$

(ii) Unitarity,

$$R_{a,b}^{c,d}(\theta) R_{c,d}^{e,f}(-\theta) = \rho(\theta)\delta_{a,e}\delta_{b,f} \tag{2.6a}$$

$$R^{*\gamma,\delta}_{\alpha,\beta}(\theta) R^{*\mu,\nu}_{\gamma,\delta}(-\theta) = \rho^*(\theta)\delta_{\alpha,\mu}\delta_{\beta,\nu}. \tag{2.6b}$$

(iii) Regularity,

$$R(0) = c_0 I. \tag{2.7}$$

(iv) A matrix M exists such that

$$R_{a,b}^{c,d}(\theta)M_{b,e}R_{f,d}^{g,e}(-\theta - 2\eta)M_{f,h}^{-1} \propto \delta_{a,g}\delta_{c,h}. \tag{2.8}$$

(v) The t-matrices verify,

$$[\bar{t}_{a,b}(\theta)]_{\alpha,\beta}[\tilde{t}_{\beta,\gamma}(-\theta)]_{b,c} = \tilde{\rho}(\theta)\delta_{a,c}\delta_{\alpha,\gamma}. \tag{2.9}$$

We consider a non-homogeneous chain with $2N$ sites in which the site spaces are alternating in the representations {3} and {3*}. This chain has associated the operator

$$T_{a,b}^{(\text{alt})}(\theta, \alpha) = t_{a,a_1}^{(1)}(\theta)\bar{t}_{a_1,a_2}^{(2)}(\theta + \alpha) \dots t_{a_{2N-2},a_{2N-1}}^{(2N-1)}(\theta)\bar{t}_{a_{2N-1},b}^{(2N)}(\theta + \alpha) \tag{2.10}$$

which is a matrix in the auxiliary space called monodromy matrix, since it describes the transportation along the chain. The elements of this matrix are operators on the space tensor product of the site spaces. It is graphically represented in figure 3.

Since the t and $\bar{t}$ matrices fulfil (2.3), $T^{(\text{alt})}$ also verifies the YBE

$$R(\theta - \theta')[T^{(\text{alt})}(\theta) \otimes T^{(\text{alt})}(\theta')] = [T^{(\text{alt})}(\theta') \otimes T^{(\text{alt})}(\theta)]R(\theta - \theta'). \tag{2.11}$$

Following the standard procedure, we take the transfer matrices

$$\tau^{(\text{alt})}(\theta, \alpha) = T_{a,a}^{(\text{alt})}(\theta, \alpha) \tag{2.12}$$

$$(h^{(1)}_{i,i+1})_{a,\beta;\,b,\gamma} = [\,\dot{\bar{t}}_{a,c}(\alpha)\,]_{\beta,\delta}\,[\,\tilde{t}_{\delta,\gamma}(-\alpha)\,]_{c,b} =$$

(a)

$$(h^{(2)}_{i,i+1,i+2})_{a,\beta,c;\,b,\gamma,d} = [\,\bar{t}_{a,e}(\alpha)\,]_{\beta,\delta}[\,\dot{t}_{e,d}(0)\,]_{c,f}[\,\tilde{t}_{\delta,\gamma}(-\alpha)\,]_{f,b} =$$
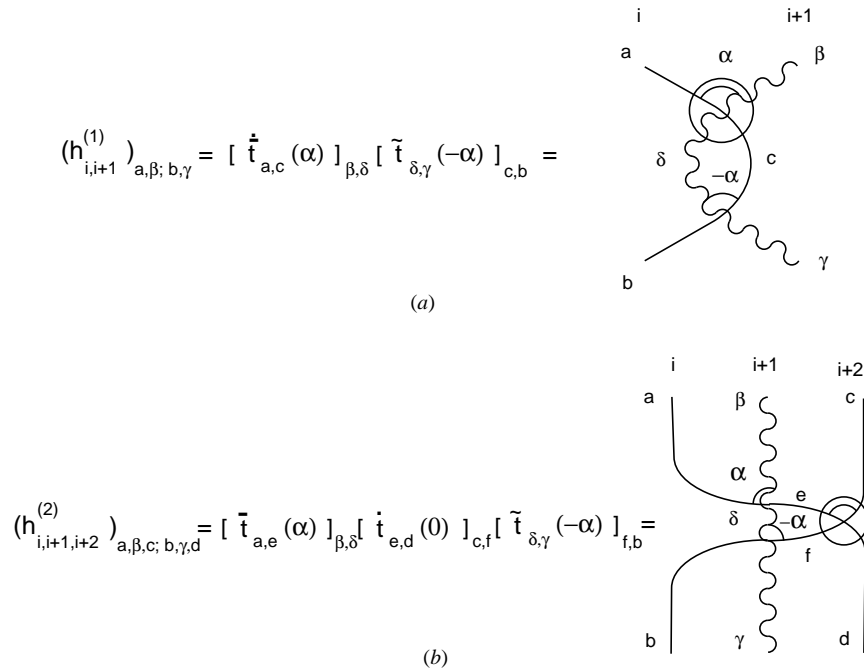
(b)

Figure 4.

which are the trace of the monodromy matrices. Due to (2.11), the operators corresponding to different values of the argument θ commute,

$$[\tau^{(\mathrm{alt})}(\theta, \alpha), \tau^{(\mathrm{alt})}(\theta', \alpha)] = 0. \tag{2.13}$$

The successive derivatives of the transfer matrix at $\theta = 0$ give us a family of commuting operators that describe a solvable system, the Hamiltonian of that system being the first derivative,

$$H = \frac{\mathrm{d}}{\mathrm{d}\theta}\ln \tau^{(\mathrm{alt})}(\theta, \alpha)|_{\theta=0}. \tag{2.14}$$

In a homogeneous chain the Hamiltonian is a sum of nearest-neighbour interactions terms (two-site operators). In our case, it is very different due to inhomogeneities and there are also next-to-nearest-neighbour interaction terms (three-site operators). Collecting separately the two kinds of terms, the Hamiltonian becomes

$$H = \frac{1}{\tilde{\rho}(\alpha)} \sum_{\substack{i=1 \\ i=\mathrm{odd}}}^{2N-1} h^{(1)}_{i,i+1} + \frac{1}{c_0\tilde{\rho}(\alpha)} \sum_{\substack{i=1 \\ i=\mathrm{odd}}}^{2N-1} h^{(2)}_{i,i+1,i+2} \tag{2.15}$$

with

$$(h^{(1)}_{i,i+1})_{a,\beta;b,\gamma} = [\dot{\bar{t}}_{a,c}(\alpha)]_{\beta,\delta}[\tilde{t}_{\delta,\gamma}(-\alpha)]_{c,b} \tag{2.16}$$

and

$$(h^{(2)}_{i,i+1,i+2})_{a,\beta,c;b,\gamma,d} = [\bar{t}_{a,e}(\alpha)]_{\beta,\delta}[\dot{t}_{e,d}(0)]_{c,f}[\tilde{t}_{\delta,\gamma}(-\alpha)]_{f,b} \tag{2.17}$$

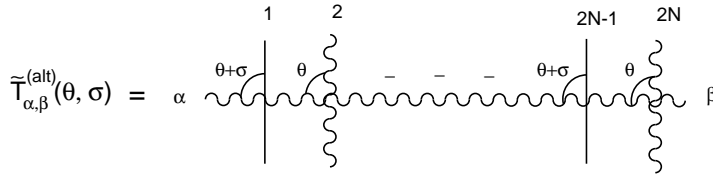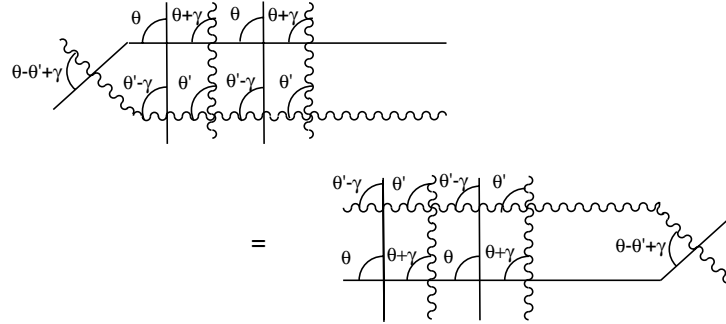that are graphically expressed in figures 4(a) and (b) respectively.

Figure 5.

Figure 6.

A similar process can be made by using as auxiliary space the σ one. Thus, we define the new monodromy matrix

$$\tilde{T}_{\alpha,\beta}^{(\text{alt})}(\theta,\sigma) = \tilde{t}_{\alpha,\alpha_1}^{(1)}(\theta+\sigma)t_{\alpha_1,\alpha_2}^{*(2)}(\theta)\ldots\tilde{t}_{\alpha_{2N-2},\alpha_{2N-1}}^{(2N-1)}(\theta+\sigma)t_{\alpha_{2N-1},\beta}^{*(2N)}(\theta) \qquad (2.18)$$

graphically represented in figure 5. It fulfills the YBE

$$R^*(\theta-\theta')[\tilde{T}^{(\text{alt})}(\theta-\sigma)\otimes\tilde{T}^{(\text{alt})}(\theta'-\sigma)] = [\tilde{T}^{(\text{alt})}(\theta'-\sigma)\otimes\tilde{T}^{(\text{alt})}(\theta-\sigma)]R^*(\theta-\theta').$$

$$(2.19)$$

The Hamiltonian obtained from this monodromy matrix using a formula similar to (2.12) is

$$\tilde{H} = \frac{1}{\tilde{\rho}(\sigma)}\sum_{\substack{i=2\\i=\text{even}}}^{2N}\tilde{h}_{i,i+1}^{(1)} + \frac{1}{c_0\tilde{\rho}(\sigma)}\sum_{\substack{i=2\\i=\text{even}}}^{2N}\tilde{h}_{i,i+1,i+2}^{(2)} \qquad (2.20)$$

with

$$(\tilde{h}_{i,i+1}^{(1)})_{\alpha,a;\beta,b} = [\dot{\tilde{t}}_{\alpha,\delta}(\sigma)]_{a,c}[\bar{t}_{c,b}(-\sigma)]_{\delta,\beta} \qquad (2.21)$$

and

$$(\tilde{h}_{i,i+1,i+2}^{(2)})_{\alpha,a,\mu;\beta,b,\nu} = [\tilde{t}_{\alpha,\delta}(\sigma)]_{a,c}[\dot{t}_{\delta,\nu}^{*}(0)]_{\mu,\rho}[\bar{t}_{c,b}(-\sigma)]_{\rho,\beta}. \qquad (2.22)$$

The monodromy matrices $T^{(\text{alt})}$ and $\tilde{T}^{(\text{alt})}$ fulfil the following YBE

$$[\bar{t}_{a,b}(\theta-\theta'+\gamma)]_{\alpha,\beta}T_{b,c}^{(\text{alt})}(\theta,\gamma)\tilde{T}_{\beta,\delta}^{(\text{alt})}(\theta',-\gamma)$$
$$= \tilde{T}_{\alpha,\mu}^{(\text{alt})}(\theta',-\gamma)T_{a,d}^{(\text{alt})}(\theta,\gamma)[\bar{t}_{d,c}(\theta-\theta'+\gamma)]_{\mu,\delta} \qquad (2.23)$$

that is graphically expressed in figure 6.

As a consequence of (2.23) the transfer matrices $\tau^{(\text{alt})}$ and $\tilde{\tau}^{(\text{alt})}$ commute

$$[\tau^{(\text{alt})}(\theta,\alpha),\tilde{\tau}^{(\text{alt})}(\theta',-\alpha)] = 0 \qquad (2.24)$$

and then, the derived Hamiltonians H and $\tilde{H}$ also commute. Thus, both can be simultaneously diagonalized with common eigenstates.

3. Quantum chain with the site states alternating in two representations of su(3)

In this section we describe a non-homogeneous chain that we form by alternating two representations of su(3). We denote a representation by the indices of its associated Dynkin diagram (m_1, m_2). The vector space s is taken as the representation $(1, 0) \equiv \{3\}$ and the space σ is the generic representation (m_1, m_2).

The t operator acting on the site and auxiliary spaces, both s, can be written [27]

$$t(\theta, \gamma) = \begin{pmatrix} \frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \lambda \frac{(q^{-1}-q)}{2} f_1 & \lambda^{-1} \frac{(q^{-1}-q)}{2}[f_2, f_1] \\ \lambda^{-1} \frac{(q^{-1}-q)}{2} e_1 & \frac{1}{2}(\lambda^3 q^{-N^\beta} - \lambda^{-3} q^{N^\beta}) & \lambda \frac{(q^{-1}-q)}{2} f_2 \\ \lambda \frac{(q^{-1}-q)}{2}[e_1, e_2] & \lambda^{-1} \frac{(q^{-1}-q)}{2} e_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma} - \lambda^{-3} q^{N^\gamma}) \end{pmatrix} \tag{3.1}$$

where the parameters λ and q have been taken as the functions of θ and γ

$$\lambda = e^{\frac{\theta}{2}} \qquad q = e^{-\gamma} \tag{3.2}$$

and the N-matrices are

$$N^\alpha = \tfrac{2}{3}h_1 + \tfrac{1}{3}h_2 + \tfrac{1}{3}I \tag{3.3a}$$

$$N^\beta = -\tfrac{1}{3}h_1 + \tfrac{1}{3}h_2 + \tfrac{1}{3}I \tag{3.3b}$$

$$N^\gamma = -\tfrac{1}{3}h_1 - \tfrac{2}{3}h_2 + \tfrac{1}{3}I \tag{3.3c}$$

where $\{e_i, f_i, q^{\pm h_i}\}$, $i = 1, 2$, are the Cartan generators of the deformed algebra $U_q(\mathrm{sl}(3))$.

To obtain the operators $\bar{t}(\theta, \gamma)$, we take (3.1) as a basis and write

$$\bar{t}(\lambda) = \begin{pmatrix} \frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \lambda \frac{(q^{-1}-q)}{2} F_1 & \lambda^{-1} \frac{(q^{-1}-q)}{2} F_3 \\ \lambda^{-1} \frac{(q^{-1}-q)}{2} E_1 & \frac{1}{2}(\lambda^3 q^{-N^\beta} - \lambda^{-3} q^{N^\beta}) & \lambda \frac{(q^{-1}-q)}{2} F_2 \\ \lambda \frac{(q^{-1}-q)}{2} E_3 & \lambda^{-1} \frac{(q^{-1}-q)}{2} E_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma} - \lambda^{-3} q^{N^\gamma}) \end{pmatrix} \tag{3.4}$$

where the operators $\{E_i, F_i\}$, $i = 1, 3$, are unknown and will be determined by imposing the YBE,

$$R(\theta - \theta', \gamma) \cdot [\bar{t}(\theta, \gamma) \otimes \bar{t}(\theta', \gamma)] = [\bar{t}(\theta', \gamma) \otimes \bar{t}(\theta, \gamma)] \cdot R(\theta - \theta', \gamma) \tag{3.5}$$

that is shown in figure 2(d). The $R^{b,d}_{c,a}(\theta) \equiv [t_{a,b}(\theta, \gamma)]_{c,d}$ is given by

$$R(\lambda, \mu) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \tag{3.6}$$

with

$$a(\lambda, \mu) = \tfrac{1}{2}(\lambda^3 \mu^{-3} q^{-1} - \lambda^{-3} \mu^3 q) \tag{3.7a}$$

$$b(\lambda, \mu) = \tfrac{1}{2}(\lambda^3 \mu^{-3} - \lambda^{-3} \mu^3) \tag{3.7b}$$

$$c(\lambda, \mu) = \tfrac{1}{2}(q^{-1} - q)\lambda \mu^{-1} \tag{3.7c}$$

$$d(\lambda, \mu) = \tfrac{1}{2}(q^{-1} - q)\lambda^{-1} \mu. \tag{3.7d}$$

The relations obtained are

$$E_1 q^{N^\alpha} = q^{-1} q^{N^\alpha} E_1 \tag{3.8a}$$

$$E_1 q^{N^\beta} = q q^{N^\beta} E_1 \tag{3.8b}$$

$$F_1 q^{N^\alpha} = q q^{N^\alpha} F_1 \tag{3.8c}$$

$$F_1 q^{N^\beta} = q^{-1} q^{N^\beta} F_1 \tag{3.8d}$$

$$E_2 q^{N^\alpha} = q q^{N^\alpha} E_2 \tag{3.8e}$$

$$E_2 q^{N^\beta} = q^{-1} q^{N^\beta} E_2 \tag{3.8f}$$

$$F_2 q^{N^\alpha} = q^{-1} q^{N^\alpha} F_2 \tag{3.8g}$$

$$F_2 q^{N^\beta} = q q^{N^\beta} F_2 \tag{3.8h}$$

$$[E_1, F_1] = (q^{-1} - q)(q^{N^\beta - N^\alpha} - q^{N^\alpha - N^\beta}) \tag{3.8i}$$

$$(q^{N^\gamma - N^\beta} - q^{N^\beta - N^\gamma}) \tag{3.8j}$$

$$E_3 = \frac{1}{(q^{-1} - q)} q^{-N^\beta} [E_1, E_2] \tag{3.8k}$$

$$F_3 = \frac{1}{(q^{-1} - q)} q^{N^\beta} [F_2, F_1] \tag{3.8l}$$

and besides, the modified Serre relations

$$q^{-1} E_1 E_1 E_2 - (q + q^{-1}) E_1 E_2 E_1 + q E_2 E_1 E_1 = 0 \tag{3.9a}$$

$$q E_2 E_2 E_1 - (q + q^{-1}) E_2 E_1 E_2 + q^{-1} E_1 E_2 E_2 = 0 \tag{3.9b}$$

$$q^{-1} F_1 F_1 F_2 - (q + q^{-1}) F_1 F_2 F_1 + q F_2 F_1 F_1 = 0 \tag{3.9c}$$

$$q F_2 F_2 F_1 - (q + q^{-1}) F_2 F_1 F_2 + q^{-1} F_1 F_2 F_2 = 0 \tag{3.9d}$$

should be verified. It must be noted that relations (3.8a)–(3.8l) are the usual ones for the quantum group $U_q(\mathrm{sl}(3))$ while relations (3.9a)–(3.9d) are not the usual ones for the stated group, and because of this, the YBE is not verified if the generators e_i and f_i, pertaining to the deformed algebra, are taken as E_i and F_i. This induces us to take

$$F_i = \tfrac{1}{2}(q^{-1} - q) Z_i f_i \tag{3.10a}$$

$$E_i = \tfrac{1}{2}(q^{-1} - q) e_i Z_i^{-1} \qquad i = 1, 2 \tag{3.10b}$$

where e_i and f_i, $i = 1, 2$, are the generators of $U_q(\mathrm{sl}(3))$ in the representation (m_1, m_2) and Z_i are two diagonal operators that were obtain by imposing the verification of the relations (3.8a)–(3.8l) and (3.9a)–(3.9d). In this way, one obtains the general form of these operators given by

$$Z_1 = q^{a_1 h_1 - \frac{1}{3} h_2 + a_3 I} \tag{3.11a}$$

$$Z_2 = q^{\frac{1}{3} h_1 + (a_1 + \frac{1}{3}) h_2 + b_3 I}. \tag{3.11b}$$

The knowledge of the operator $\bar{t}$ permits us to build the monodromy operator of any multistate chain that mixes two representations. As an example, for the chain that mixes the $\{3\}$ and the (m_1, m_2) representations the monodromy operator is

$$T_{a,b}^{(\mathrm{alt})}(\theta) = t_{a,a_1}^{(1)}(\theta) \bar{t}_{a_1,a_2}^{(2)}(\theta) \dots t_{a_{2N-2},a_{2N-1}}^{(2N-1)}(\theta) \bar{t}_{a_{2N-1},b}^{(2N)}(\theta) \tag{3.12}$$

that is represented graphically as shown in figure 3.

4. Bethe ansatz equations of the models with space sites in different representations of su(3)

In this section we are going to solve an alternating chain that mixes the {3} and {3*} representations of su(3) and the results will be generalized to chains that mix two arbitrary representations.

In this case, the t operator is given by (3.1) that can be written in matrix form

$$
t(\theta) = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\
0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{pmatrix}
\tag{4.1}
$$

with

$$a(\theta) = \sinh\left(\tfrac{3}{2}\theta + \gamma\right) \tag{4.2a}$$

$$b(\theta) = \sinh\left(\tfrac{3}{2}\theta\right) \tag{4.2b}$$

$$c(\theta) = \sinh(\gamma)\mathrm{e}^{\frac{\theta}{2}} \tag{4.2c}$$

$$d(\theta) = \sinh(\gamma)\mathrm{e}^{\frac{-\theta}{2}}. \tag{4.2d}$$

In the same way, $\bar{t}$ is obtained from (3.4) by taking in (3.10a), (3.10b) the generators of su(3) in the {3*} representation,

$$
e_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad
e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}
$$
$$
f_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad
f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\tag{4.3}
$$

Besides, we must fix in (3.11a), (3.11b) the values of a_1, a_3 and b_3. By taking

$$a_1 = \tfrac{2}{9} \qquad a_3 = 0 \qquad b_3 = 0 \tag{4.4}$$

and rescaling θ by

$$\theta = \theta + \tfrac{5}{9}\gamma \tag{4.5}$$

we find

$$
\bar{t}(\theta) = \begin{pmatrix}
\bar{a} & 0 & 0 & 0 & \bar{c} & 0 & 0 & 0 & \bar{d} \\
0 & \bar{b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{b} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{b} & 0 & 0 & 0 & 0 & 0 \\
\bar{d} & 0 & 0 & 0 & \bar{a} & 0 & 0 & 0 & \bar{c} \\
0 & 0 & 0 & 0 & 0 & \bar{b} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{b} & 0 \\
\bar{c} & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & \bar{a}
\end{pmatrix}
\tag{4.6}
$$

with

$$\bar{a}(\theta) = \sinh\left(\frac{3}{2}\theta + \frac{\gamma}{2}\right) \tag{4.7a}$$

$$\bar{b}(\theta) = \sinh\left(\frac{3}{2}(\theta + \gamma)\right) \tag{4.7b}$$

$$\bar{c}(\theta) = -\sinh(\gamma)e^{\frac{(\theta+\gamma)}{2}} \tag{4.7c}$$

$$\bar{d}(\theta) = -\sinh(\gamma)e^{\frac{-(\theta+\gamma)}{2}}. \tag{4.7d}$$

As we take the {3*} representation as auxiliary space, the *R*-matrix is

$$R(\theta) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \tag{4.8}$$

and in this case the matrix *M* defined in (2.8) is the identity and the functions defined in equations (2.6a), (2.6b) and (2.7) are

$$\rho(\theta) = \rho^*(\theta) = \sinh\left(\gamma - \tfrac{3}{2}\theta\right)\sinh\left(\gamma + \tfrac{3}{2}\theta\right) \tag{4.9a}$$

$$c_0 = c_0^* = \sinh\gamma \tag{4.9b}$$

$$\tilde{\rho}(\theta) = \tfrac{1}{2}\left(\cosh(3\gamma) - \cosh(3\theta)\right). \tag{4.9c}$$

We group two neighbour sites in the chain and form the operator

$$\hat{t}_{a,b}^{(i,i+1)}(\theta, \alpha) = t_{a,a_1}^{(i)}(\theta)\bar{t}_{a_1,b}^{(i+1)}(\theta + \alpha) \qquad i \text{ odd.} \tag{4.10}$$

The monodromy matrix that corresponds to this model

$$T_{a,b}^{(\text{alt})}(\theta, \alpha) = \hat{t}_{a,a_1}^{(1,2)}(\theta, \alpha)\hat{t}_{a_1,a_2}^{(3,4)}(\theta, \alpha)\dots\bar{t}_{a_{2N-1},b}^{(2N-1,2N)}(\theta, \alpha). \tag{4.11}$$

This operator can be written in the auxiliary space as a matrix

$$T^{\text{alt}}(\theta, \alpha) = \begin{pmatrix} A(\theta, \alpha) & B_2(\theta, \alpha) & B_3(\theta, \alpha) \\ C_2(\theta, \alpha) & D_{2,2}(\theta, \alpha) & D_{2,3}(\theta, \alpha) \\ C_3(\theta, \alpha) & D_{3,2}(\theta, \alpha) & D_{3,3}(\theta, \alpha) \end{pmatrix} \tag{4.12}$$

whose elements are operators in the tensorial product of the site spaces,

$$S = \bigotimes_{i=\text{odd}} s_{i,i+1} \tag{4.13}$$

$s_{i,i+1}$ being the tensorial product of site spaces (*i*) and (*i* + 1) and isomorphic to the {3} and {3*} representation product

$$s_{i,i+1} = s_i \otimes s_{i+1} \sim \{3\} \otimes \{3^*\}. \tag{4.14}$$

The YBE for $T^{(\text{alt})}$ can be written in terms of its components

$$B(\theta) \otimes B(\theta') = R^{(2)}(\theta - \theta') \cdot (B(\theta') \otimes B(\theta)) = (B(\theta') \otimes B(\theta)) \cdot R^{(2)}(\theta - \theta') \tag{4.15a}$$

$$A(\theta)B(\theta') = g(\theta' - \theta)B(\theta')A(\theta) - B(\theta)A(\theta') \cdot \tilde{r}^{(2)}(\theta' - \theta) \tag{4.15b}$$

$$D(\theta) \otimes B(\theta') = g(\theta - \theta')(B(v) \otimes D(\theta)) \cdot R^{(2)}(\theta - \theta') - B(\theta) \otimes (r^{(2)}(\theta - \theta') \cdot D(\theta')) \tag{4.15c}$$

where

$$R^{(2)}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & \frac{c}{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad r^{(2)}(\theta) = \begin{pmatrix} h_- & 0 \\ 0 & h_+ \end{pmatrix} \qquad \tilde{r}^{(2)}(\theta) = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}$$

(4.16)

and

$$g(\theta) = \frac{a(\theta)}{b(\theta)} \qquad h_+(\theta) = \frac{c(\theta)}{b(\theta)} \qquad h_-(\theta) = \frac{d(\theta)}{b(\theta)}.$$

(4.17)

For the site states, we use the notation

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \bar{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \bar{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(4.18)

In order to find the eigenvectors and eigenvalues of

$$\tau^{(\text{alt})}(\theta) = A(\theta) + D_{2,2}(\theta) + D_{3,3}(\theta)$$

(4.19)

we find inspiration in the NBA method and look for an eigenstate of A that serves as a pseudovacuum. For this purpose, we build the subspace of $s_{(i,i+1)}$ generated by the vectors $|u, \bar{s}\rangle$ and $|u, \bar{d}\rangle$, that we call w_i, and then the subspace

$$\Omega = w_1 \otimes w_3 \otimes \cdots \otimes w_N$$

(4.20)

of the total space of states of a chain with $2N$ sites.

In a non-homogeneous chain, we do not have a state $\|v\rangle$ such that

$$D_{i,j}\|v\rangle \propto \delta_{i,j}\|v\rangle.$$

(4.21)

For this reason, the NBA method cannot be used. Our method, instead, starts with a state $\|1\rangle \in \Omega$ verifying

$$A(\theta)\|1\rangle = [a(\theta)]^{N_3}[\bar{b}(\theta)]^{N_3^*}\|1\rangle \qquad\qquad\qquad (4.22a)$$

$$B_i\|1\rangle \neq 0 \qquad i = 2, 3 \qquad\qquad\qquad (4.22b)$$

$$C_i\|1\rangle = 0 \qquad i = 2, 3 \qquad\qquad\qquad (4.22c)$$

$$D_{i,j}\|1\rangle \in \Omega \qquad i, j = 2, 3 \qquad\qquad\qquad (4.22d)$$

N_3 (N_3^*) being the number of sites in the representation $\{3\}$ ($\{3^*\}$). In order to simplify the exposition of our method, we take $N_3 = N_3^* = N$.

Following the steps inspired in the NBA, we apply r-times the B operators to $\|1\rangle$ and build the state

$$\Psi(\boldsymbol{\mu}) \equiv \Psi(\mu_1, \ldots, \mu_r) = B_{i_1}(\mu_1) \ldots B_{i_r}(\mu_r)X_{i_1,\ldots,i_r}\|1\rangle \equiv B(\mu_1) \otimes \cdots \otimes B(\mu_r)X\|1\rangle$$

(4.23)

$X_{i_1,\ldots,i_r}$ being a r-tensor that, together with the values of the spectral parameters $\mu_1, \ldots, \mu_r$, will be determined at the end.

The action of $A(\mu)$ and $D_{i,i}(\mu)$ on Ψ is found by pushing them to the right through the $B_{i_j}(\mu_j)$'s using the commutations rules (4.15b), (4.15c). Two types of terms arise when A and $D_{i,j}$ pass through B's: the wanted and unwanted terms, similar to obtained in the NBA

method. The first one comes from the first terms of (4.15b), (4.15c). In this type of terms the A or $D_{i,i}$ and the B's keep their original arguments and give a state proportional to Ψ. The terms coming from the second terms in (4.15b), (4.15c) are called unwanted since they contain $B_i(\mu)$ and so they never give a state proportional to Ψ; so, they must cancel each other out when we sum the trace of T^{alt}. The wanted term obtained by application of A is

$$[a(\mu)]^{N_3}[\bar{b}(\mu)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - \mu) B_{i_1}(\mu_1) \ldots B_{i_r}(\mu_r) X_{i_1,\ldots,i_r} \| 1\rangle \tag{4.24}$$

and the kth unwanted term

$$- [a(\mu_k)]^{N_3}[\bar{b}(\mu_k)]^{N_3^*} \prod_{\substack{j=1 \\ j \neq k}}^{r} g(\mu_j - \mu_k)(B(u)\tilde{r}^{(2)}(\mu_k - u)) \otimes B(\mu_{k+1}) \otimes \cdots \otimes B(\mu_r)$$

$$\otimes B(\mu_1) \otimes B(\mu_{k-1}) M^{(k-1)} X \| 1\rangle \tag{4.25}$$

M being the operator arising by repeated application of (4.15a),

$$B(\mu_1) \otimes \cdots \otimes B(\mu_r) = B(\mu_{k+1}) \otimes \cdots B(\mu_r) \otimes B(\mu_1) \cdots \otimes B(\mu_{k-1}) M^{(k-1)}. \tag{4.26}$$

The application of the operators $D_{i,j}(\mu)$ to the state $\Psi(\mu)$ is a little more laborious but straightforward. The wanted term results

$$[D_{k,j}(\mu) B_{i_1}(\mu_1) \ldots B_{i_r}(\mu_r) X_{i_1,\ldots,i_r} \| 1\rangle]_{\text{wanted}}$$

$$= \prod_{i=1}^{r} g(\mu - \mu_i) B_{j_1}(\mu_1) \ldots B_{j_r}(\mu_r) R_{j_r,a_r}^{(2)a_{r-1},i_r}(\mu - \mu_r) \ldots R_{j_2,a_2}^{(2)a_1,i_2}(\mu - \mu_2)$$

$$\times R_{j_1,a_1}^{(2)j,i_1}(\mu - \mu_1) D_{k,a_r} X_{i_1,\ldots,i_r} \| 1\rangle \tag{4.27}$$

where the $R^{(2)}$'s product is taken in the auxiliary space and has the form

$$\Phi(\mu, \boldsymbol{\mu})_{a_r,j} \equiv R_{j_r,a_r}^{(2)a_{r-1},i_r} \cdots R_{j_2,a_2}^{(2)a_1,i_2} \cdot R_{j_1,a_1}^{(2)j,i_1} = \begin{pmatrix} \alpha(\mu, \boldsymbol{\mu}) & \beta(\mu, \boldsymbol{\mu}) \\ \gamma(\mu, \boldsymbol{\mu}) & \delta(\mu, \boldsymbol{\mu}) \end{pmatrix}. \tag{4.28}$$

The action of $D_{k,j}$ with $k \neq j$ on $\| 1\rangle$ is not zero. This is the main difference with the models that can be solved by NBA. Then, we try to diagonalize the matrix product

$$F(\mu, \boldsymbol{\mu}) = D(\mu) \cdot \Phi(\mu, \boldsymbol{\mu}) = \begin{pmatrix} A^{(2)}(\mu, \boldsymbol{\mu}) & B^{(2)}(\mu, \boldsymbol{\mu}) \\ C^{(2)}(\mu, \boldsymbol{\mu}) & D^{(2)}(\mu, \boldsymbol{\mu}) \end{pmatrix}. \tag{4.29}$$

By taking the terms in (4.27) with $k = j$ and adding them for $k = 2$ and 3, we obtain the wanted term

$$\prod_{j=1}^{r} g(\mu - \mu_j) B_{i_1}(\mu_1) \ldots B_{i_r}(\mu_r) \tau_{(2)}(\mu, \boldsymbol{\mu}) X_{i_1,\ldots,i_r} \| 1\rangle \tag{4.30}$$

where

$$\tau_{(2)}(\mu, \boldsymbol{\mu}) = \text{tr}(F) = A^{(2)}(\mu, \boldsymbol{\mu}) + D^{(2)}(\mu, \boldsymbol{\mu}). \tag{4.31}$$

In the same form, the kth unwanted term results

$$- \prod_{\substack{j=1 \\ j \neq k}}^{r} g(\mu_k - \mu_j)(B(\mu)r^{(2)}(\mu - \mu_k)) \otimes B(\mu_{k+1}) \otimes \cdots \otimes B(\mu_r)$$

$$\otimes B(\mu_1) \otimes B(\mu_{k-1}) M^{(k-1)} \tau_{(2)}(\mu_k, \boldsymbol{\mu}) X \| 1\rangle. \tag{4.32}$$

The sum of the wanted terms and the cancellation of the unwanted terms give us the relations

$$\tau_{(2)}(\mu, \boldsymbol{\mu}) X \| 1\rangle = \Lambda_{(2)}(\mu, \boldsymbol{\mu}) X \| 1\rangle \tag{4.33}$$

and

$$\Lambda_{(2)}(\mu_k, \boldsymbol{\mu}) = [a(\mu_k)]^{N_3} [\bar{b}(\mu_k)]^{N_3^*} \prod_{\substack{j=1 \\ j \neq k}}^{r} \frac{g(\mu_j - \mu_k)}{g(\mu_k - \mu_j)}. \tag{4.34}$$

We must now diagonalize (4.33).

The state $\|1\rangle \in \Omega$ and the tensor $X_{i_1,\ldots,i_r}$, $(i_j = 2, 3)$ lies in a space with 2^r dimensions, tensorial product of r two-dimensional spaces $C_l, l = 1 \ldots r$, generated by the vectors

$$e_l^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_l, \qquad e_l^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_l, \qquad l = 1 \ldots r. \tag{4.35}$$

Then, the vector $X\|1\rangle$ yields in a space $\Omega^{(2)}$ with 2^{r+N} dimensions. In this space, we take the element

$$\|1\rangle^{(2)} = e_1^1 \otimes e_2^1 \cdots \otimes e_r^1 \otimes |u\bar{s}\rangle_1 \otimes \cdots \otimes |u\bar{s}\rangle_N \tag{4.36}$$

which is annihilated by $C^{(2)}(\mu, \boldsymbol{\mu})$. (Note that the operators α, β, γ and δ of (4.28) act on the first part of $\|1\rangle^{(2)}$ and the operators $D_{i,j}$ on the second part.) The application of the operators $A^{(2)}$ and $D^{(2)}$ gives

$$A^{(2)}(\mu, \boldsymbol{\mu})\|1\rangle^{(2)} = [b(\mu)]^{N_3} [\bar{b}(\mu)]^{N_3^*} \|1\rangle^{(2)} \tag{4.37a}$$

$$D^{(2)}(\mu, \boldsymbol{\mu})\|1\rangle^{(2)} = \prod_{i=1}^{r} \frac{1}{g(\mu - \mu_i)} [a(\mu)]^{N_3} [\bar{b}(\mu)]^{N_3^*} \|1\rangle^{(2)}. \tag{4.37b}$$

The important fact is that $F(\mu, \boldsymbol{\mu})$ verifies the YBE with the $R^{(2)}$ matrix given in (4.16),

$$R^{(2)}(\mu - \mu')[F(\mu, \boldsymbol{\mu}) \otimes F(\mu', \boldsymbol{\mu})] = [F(\mu', \boldsymbol{\mu}) \otimes F(\mu, \boldsymbol{\mu})]R^{(2)}(\mu - \mu') \tag{4.38}$$

which, in a second step, permits us to solve the system. From this equation, we obtain the commutation rules

$$A^{(2)}(\mu) \cdot B^{(2)}(\mu') = g(\mu' - \mu)B^{(2)}(\mu') \cdot A^{(2)}(\mu) - h_+(\mu' - \mu)B^{(2)}(\mu) \cdot A^{(2)}(\mu') \tag{4.39a}$$

$$D^{(2)}(\mu) \cdot B^{(2)}(\mu') = g(\mu - \mu')B^{(2)}(\mu') \cdot D^{(2)}(\mu) - h_+(\mu - \mu')B^{(2)}(\mu) \cdot D^{(2)}(\mu'). \tag{4.39b}$$

In this second step, we build the vector

$$\Psi^{(2)}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = B^{(2)}(\lambda_1, \boldsymbol{\mu}) \cdots B^{(2)}(\lambda_s, \boldsymbol{\mu})\|1\rangle^{(2)}. \tag{4.40}$$

The action of $A^{(2)}(\lambda, \boldsymbol{\mu})$ on $\Psi^{(2)}$ gives the wanted term

$$[b(\lambda)]^{N_3} [\bar{b}(\lambda)]^{N_3^*} \prod_{i=1}^{s} g(\lambda_i - \lambda)B^{(2)}(\lambda_1, \boldsymbol{\mu}) \ldots B^{(2)}(\lambda_s, \boldsymbol{\mu})\|1\rangle^{(2)} \tag{4.41}$$

and the kth unwanted term

$$- h_+(\lambda_k - \lambda)[b(\lambda_k)]^{N_3} [\bar{b}(\lambda_k)]^{N_3^*} \prod_{\substack{i=1 \\ i \neq k}}^{s} g(\lambda_i - \lambda_k)$$

$$\times B^{(2)}(\lambda, \boldsymbol{\mu})B^{(2)}(\lambda_{k+1}, \boldsymbol{\mu}) \ldots B^{(2)}(\lambda_{k-1}, \boldsymbol{\mu})\|1\rangle^{(2)}. \tag{4.42}$$

In the same form, the action of $D^{(2)}(\lambda, \boldsymbol{\mu})$ on $\Psi^{(2)}$ gives the wanted term

$$[b(\lambda)]^{N_3} [\bar{a}(\lambda)]^{N_3^*} \prod_{i=1}^{s} g(\lambda - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\lambda - \mu_j)} B^{(2)}(\lambda_1, \boldsymbol{\mu}) \ldots B^{(2)}(\lambda_s, \boldsymbol{\mu})\|1\rangle^{(2)} \tag{4.43}$$

and the kth unwanted term

$$- h_-(\lambda - \lambda_k)[b(\lambda_k)]^{N_3}[\bar{a}(\lambda_k)]^{N_3^*} \prod_{\substack{i=1\\i\neq k}}^{s} g(\lambda_k - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} B^{(2)}(\lambda, \boldsymbol{\mu})$$

$$B^{(2)}(\lambda_{k+1}, \boldsymbol{\mu}) \dots B^{(2)}(\lambda_{k-1}, \boldsymbol{\mu}) \| 1 \rangle^{(2)}. \tag{4.44}$$

The cancellation of the unwanted terms and the sum of the wanted terms give us the equations

$$\left[\frac{\bar{a}(\lambda_k)}{\bar{b}(\lambda_k)} \right]^{N_3^*} \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} = \prod_{\substack{i=1\\i\neq k}}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \qquad k = 1, \dots, s \tag{4.45}$$

and

$$\Lambda_{(2)}(\mu_k, \boldsymbol{\mu}) = \prod_{i=1}^{s} g(\lambda_i - \mu_k)[b(\mu_k)]^{N_3}[\bar{a}(\mu_k)]^{N_3^*}. \tag{4.46}$$

Then, by comparing equations (4.34) and (4.46) and calling $\bar{g}(\theta) = \bar{a}(\theta)/\bar{b}(\theta)$, we obtain the coupled Bethe equations

$$[\bar{g}(\lambda_k)]^{N_3^*} = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{\substack{i=1\\i\neq k}}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \tag{4.47a}$$

$$[g(\mu_l)]^{N_3} = \prod_{\substack{j=1\\j\neq l}}^{r} \frac{g(\mu_l - \mu_j)}{g(\mu_j - \mu_l)} \prod_{i=1}^{s} g(\lambda_i - \mu_l) \tag{4.47b}$$

and the eigenvalue of the trace of $T^{(\text{alt})}$

$$\Lambda(\mu) = [a(\mu)]^{N_3}[\bar{b}(\mu)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - \mu) + [b(\mu)]^{N_3} \prod_{j=1}^{r} g(\mu - \mu_j)$$

$$\times \left[[\bar{b}(\mu)]^{N_3^*} \prod_{i=1}^{s} g(\lambda_i - \mu) + [a(\mu)]^{N_3} \prod_{i=1}^{s} g(\mu - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\mu - \mu_j)} \right] \tag{4.48}$$

that is the solution to the spectrum of our problem.

The Hamiltonian of the alternating chain can be obtained with (2.16) and (2.17). The results for $h^{(1)}$ and $h^{(2)}$ are

$$h_{i,i+1}^{(1)} = \frac{\sinh\gamma(1 + 2\cosh\gamma)}{2(\cosh(3\gamma) - 1)} \sum_{\alpha=1}^{8} J_\alpha \lambda_i^\alpha \otimes \bar{\lambda}_{i+1}^\alpha \tag{4.49}$$

and

$$h_{i,i+1,i+2}^{(2)} = \sum_{\alpha=1}^{8} m_\alpha I_i \otimes \bar{\lambda}_{i+1}^\alpha \otimes \lambda_{i+2}^\alpha + \sum_{\alpha=1}^{8} m_\alpha' \lambda_i^\alpha \otimes I_{i+1} \otimes \lambda_{i+2}^\alpha$$

$$+ k \left(\lambda_i^3 \otimes I_{i+1} \otimes \lambda_{i+2}^8 - \lambda_i^8 \otimes I_{i+1} \otimes \lambda_{i+2}^3 \right) + k' f_{i,i+1,i+2} \tag{4.50}$$

where we have used the Gell–Mann matrices λ and $\bar{\lambda}$ for the {3} and {3*} representations respectively, being the coefficients,

$$m_\alpha = \begin{cases} \dfrac{\sinh\gamma(1 + 2\cosh\gamma)}{2(\cosh(3\gamma) - 1)} & \text{if } \alpha \neq 3, 8 \\[4mm] \dfrac{\sinh\gamma(-1 + 4\cosh^2\gamma)}{2(\cosh(3\gamma) - 1)} & \text{if } \alpha = 3, 8 \end{cases} \tag{4.51a}$$

$$m'_\alpha = \begin{cases} \dfrac{\sinh^2 \frac{\gamma}{2}(1+2\cosh\gamma)(3+2\cosh\gamma)}{2\sinh\gamma(\cosh(3\gamma)-1)} & \text{if } \alpha \neq 3,8 \\[4mm] \dfrac{\sinh^2 \frac{\gamma}{2}(1+2\cosh\gamma)(3+\cosh\gamma+\cosh(2\gamma))}{2\sinh\gamma(\cosh(3\gamma)-1)} & \text{if } \alpha = 3,8 \end{cases} \tag{4.51b}$$

$$k = \frac{\sqrt{3}\sinh^2 \frac{\gamma}{2}(1+2\cosh\gamma)^2}{4(\cosh(3\gamma)-1)} \tag{4.51c}$$

$$k' = \frac{3\sinh^2 \frac{\gamma}{2}(1+2\cosh\gamma)}{\sinh\gamma(\cosh(3\gamma)-1)}. \tag{4.51d}$$

The term $f_{i,i+1,i+2}$ is

$$f_{i,i+1,i+2} = \sum_{\mu,\nu,\rho=1}^{8} d_{\mu,\nu,\rho}\left(\cosh^2\left(\frac{\gamma}{2}\right) - \frac{\sinh\gamma}{4}\epsilon_{\mu,\nu,\rho}\right)\lambda_i^\mu \otimes \bar\lambda_{i+1}^\nu \otimes \lambda_{i+2}^\rho$$

$$+ \sum_{\alpha=1}^{8}\{w_{3,\alpha}(\lambda_i^3 \otimes \bar\lambda_{i+1}^\alpha \otimes \lambda_{i+2}^\alpha - \lambda_i^\alpha \otimes \bar\lambda_{i+1}^\alpha \otimes \lambda_{i+2}^3)$$

$$+ w_{8,\alpha}(\lambda_i^8 \otimes \bar\lambda_{i+1}^\alpha \otimes \lambda_{i+2}^\alpha - \lambda_i^\alpha \otimes \bar\lambda_{i+1}^\alpha \otimes \lambda_{i+2}^8) + v_{3,\alpha}\lambda_i^\alpha \otimes \bar\lambda_{i+1}^3 \otimes \lambda_{i+2}^\alpha$$

$$+ v_{8,\alpha}\lambda_i^\alpha \otimes \bar\lambda_{i+1}^8 \otimes \lambda_{i+2}^\alpha\} + z(\lambda_i^3 \otimes \bar\lambda_{i+1}^8 \otimes \lambda_{i+2}^3 + \lambda_i^3 \otimes \bar\lambda_{i+1}^3 \otimes \lambda_{i+2}^8$$

$$+ \lambda_i^8 \otimes \bar\lambda_{i+1}^3 \otimes \lambda_{i+2}^3 - \lambda_i^8 \otimes \bar\lambda_{i+1}^8 \otimes \lambda_{i+2}^8) \tag{4.52}$$

where

$$w_3 = \frac{\sinh\gamma}{4}(2 \quad 2 \quad 0 \quad -1 \quad -1 \quad -1 \quad -1 \quad 0) \tag{4.53a}$$

$$w_8 = \frac{\sqrt{3}\sinh\gamma}{4}(0 \quad 0 \quad 0 \quad -1 \quad -1 \quad 1 \quad 1 \quad 0) \tag{4.53b}$$

$$v_3 = \frac{-\sinh^2 \frac{\gamma}{2}}{2}(0 \quad 0 \quad 0 \quad -1 \quad -1 \quad 1 \quad 1 \quad 0) \tag{4.53c}$$

$$v_8 = \frac{\sinh^2 \frac{\gamma}{2}}{2\sqrt{3}}(2 \quad 2 \quad 0 \quad -1 \quad -1 \quad -1 \quad -1 \quad 0) \tag{4.53d}$$

$$z = \frac{\sinh^2\gamma}{\sqrt{3}} \tag{4.53e}$$

$d_{\mu,\nu,\rho}$ are the totally symmetric structure constants of SU(3), and $\epsilon_{\mu,\nu,\rho}$ is the totally antisymmetric tensor.

As a first generalization, we can now apply the method to a chain that mixes the $(1,0) \equiv \{3\}$ and (m_1,m_2) representations. In this model we take again the $(1,0)$ representation as auxiliary space; then we have the same R-matrix (4.8) for the YBE.

The highest weight of the (m_1,m_2) representation is

$$\Lambda_h = \frac{2m_1 + m_2}{3}\alpha_1 + \frac{m_1 + 2m_2}{3}\alpha_2 \tag{4.54}$$

where α_1 and α_2 are the simple roots of su(3).

Through (3.3a)–(3.3c), (3.4), together with the commutation rules of su(3), it is possible to know the action of elements of the $\bar{t}$-matrix on the highest-weight vector. We obtain

$$\bar{t}_{1,1}(\theta)|\Lambda_h\rangle = \bar{a}(\theta)|\Lambda_h\rangle \tag{4.55a}$$

$$\bar{t}_{2,2}(\theta)|\Lambda_h\rangle = \bar{b}_1(\theta)|\Lambda_h\rangle \tag{4.55b}$$

$$\bar{t}_{3,3}(\theta)|\Lambda_h\rangle = \bar{b}_2(\theta)|\Lambda_h\rangle \tag{4.55c}$$

where

$$\bar{a}(\theta) = \sinh(\tfrac{3}{2}\theta + (\tfrac{2}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma) \tag{4.56a}$$

$$\bar{b}_1(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma) \tag{4.56b}$$

$$\bar{b}_2(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 - \tfrac{2}{3}m_2 + \tfrac{1}{3})\gamma). \tag{4.56c}$$

As before, we group neighbour sites and build the monodromy operator T that can be represented by a matrix in the auxiliary space as in (4.12). The two sites space is now

$$s_{i,i+1} \sim (1,0) \otimes (m_1, m_2). \tag{4.57}$$

In this space, the subspace w_i is now generated by the highest weight of the $(1,0)$ representation and the subspace V generated by the states

$$\{|\Lambda_h\rangle, f_2|\Lambda_h\rangle, f_2^2|\Lambda_h\rangle, \ldots\} \tag{4.58}$$

where f_2 is the generator of sl(3) in the (m_1, m_2) representation.

We form the subspace Ω as in (4.20) and built the state $\|1\rangle \in \Omega$ which must satisfy

$$A(\theta)\|1\rangle \propto \|1\rangle \tag{4.59a}$$

$$D_{i,i}(\theta)\|1\rangle \propto \|1\rangle \qquad i = 2,3 \tag{4.59b}$$

$$B_i\|1\rangle \neq 0 \qquad i = 2,3 \tag{4.59c}$$

$$C_i\|1\rangle = 0 \qquad i = 2,3 \tag{4.59d}$$

$$D_{i,j}\|1\rangle \in \Omega \qquad i,j = 2,3 \qquad i \neq j \tag{4.59e}$$

Then, the states $\Psi(\boldsymbol{\mu})$ analogous to (4.23) are

$$\Psi(\boldsymbol{\mu}) \equiv \Psi(\mu_1, \ldots, \mu_r) = B_{i_1}(\mu_1) \ldots B_{i_r}(\mu_r) X_{i_1,\ldots,i_r}\|1\rangle \equiv B(\mu_1) \otimes \cdots \otimes B(\mu_r) X\|1\rangle. \tag{4.60}$$

As the YBE depends on the R-matrix, we have for the new monodromy matrix the same commutations rules (4.15a)–(4.15c) as before; then we can repeat the same steps, the only difference being in the action of the operators of the monodromy matrix on the state $\|1\rangle$. The Bethe equations (BEs) that we obtain in this case are

$$[g(\mu_k)]^N [\bar{g}_1(\mu_k)]^N = \prod_{\substack{j=1 \\ j \neq k}}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) \tag{4.61a}$$

$$[\bar{g}_2(\lambda_k)]^N = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{\substack{i=1 \\ i \neq k}}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \tag{4.61b}$$

where μ_i, $i = 1, \ldots, r$, and λ_j, $j = 1, \ldots, s$, are the roots of the ansatz, the function g is given in (4.17), and

$$\bar{g}_1(\theta) = \frac{\bar{a}(\theta)}{\bar{b}_1(\theta)} \tag{4.62a}$$

$$\bar{g}_2(\theta) = \frac{\bar{b}_2(\theta)}{\bar{b}_1(\theta)}. \tag{4.62b}$$

The procedure can be generalized to chains that mix non-fundamental representations (m_1, m_2) and (m'_1, m'_2), irrespective of the number of sites and their distribution in the representations. For this purpose, it is necessary to build the monodromy matrix following an analogous process to used before. If we use a broken line for the representation (m'_1, m'_2), the monodromy matrix $T^{\text{gen}}(\theta)$ can be represented graphically as shown in figure 7.
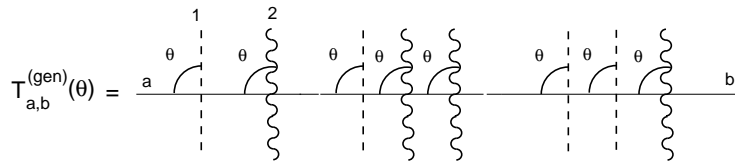
Figure 7.

By calling N_1 and N_2 the number of sites in the representations (m_1, m_2) and (m'_1, m'_2) respectively, we find the BE for this general chain

$$[\tilde{g}_1(\mu_k)]^{N_1}[\bar{g}_1(\mu_k)]^{N_2} = \prod_{\substack{j=1 \\ j \neq k}}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) \qquad (4.63a)$$

$$[\tilde{g}_2(\lambda_k)]^{N_1}[\bar{g}_2(\lambda_k)]^{N_2} = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{\substack{i=1 \\ i \neq k}}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \qquad (4.63b)$$

where $\bar{g}_1$ and $\bar{g}_2$ are given in (4.62a)–(4.62b), and $\tilde{g}_1$ and $\tilde{g}_2$ are the same as the previous ones with (m_1, m_2) replaced by (m'_1, m'_2).

In the light of this, the generalization for the case of mixed chains with more than two different representations seems simple, although the physical models that they represent will be less local and the interaction more complex.

Also we can conjecture about the solution of a non-homogeneous chain combining different representations of su(n), each representation introduces $(n-1)$ functions g_i similar to (4.62a)–(4.62b) (that we call source functions). The BEs are obtained by applying our modified Bethe ansatz (MBA) with $(n - 1)$ steps, then each solution will have a set of $(n - 1)$ equations (the same number of dots in its Dynkin diagram). The first member of the equations will be a product of the respective source functions powered to the number of sites of each representation and the second a product of g functions coming from the YBE similar to (4.63a)–(4.63b).

5. Thermodynamic limits of solutions and analysis of Bethe equations

In this section we are going to discuss the solutions of the $\{3\}$–$\{3^*\}$ model given by equations (4.46) in the limit for very large N. For that discussion, it is convenient to set the parametrization of the spectral parameters

$$\frac{3}{2}\mu_j = iv_j^{(1)} - \frac{\gamma}{2} \qquad (5.1a)$$

$$\frac{3}{2}\lambda_j = iv_j^{(2)} - \gamma \qquad (5.1b)$$

and $N = N_3 + N_3^*$ the length of the chain.

Using such parametrization, Bethe equations (4.47a)–(4.47b) can be written

$$\left[\frac{\sin(v_k^{(2)} + i\frac{\gamma}{2})}{\sin(v_k^{(2)} - i\frac{\gamma}{2})} \right]^{N_3^*} = -\prod_{j=1}^{r} \frac{\sin(v_k^{(2)} - v_j^{(1)} - i\frac{\gamma}{2})}{\sin(v_k^{(2)} - v_j^{(1)} + i\frac{\gamma}{2})} \prod_{\substack{i=1 \\ i \neq k}}^{s} \frac{\sin(v_i^{(2)} - v_k^{(2)} - i\gamma)}{\sin(v_i^{(2)} - v_k^{(2)} + i\gamma)} \qquad (5.2a)$$

$$\left[\frac{\sin{(v_k^{(1)} - \mathrm{i}\frac{\gamma}{2})}}{\sin{(v_k^{(1)} + \mathrm{i}\frac{\gamma}{2})}}\right]^{N_3} = -\prod_{\substack{j=1 \\ j \neq k}}^{r} \frac{\sin{(v_k^{(1)} - v_j^{(1)} - \mathrm{i}\gamma)}}{\sin{(v_k^{(1)} - v_j^{(1)} + \mathrm{i}\gamma)}} \prod_{i=1}^{s} \frac{\sin{(v_i^{(2)} - v_k^{(1)} - \mathrm{i}\frac{\gamma}{2})}}{\sin{(v_i^{(2)} - v_k^{(1)} + \mathrm{i}\frac{\gamma}{2})}}. \tag{5.2b}$$

In this regime, the roots must be considered in the interval $(-\pi/2, \pi/2)$. Then, we define the function

$$\phi(\chi, \alpha) = \mathrm{i} \ln \frac{\sin{(\chi + \mathrm{i}\alpha)}}{\sin{(\chi - \mathrm{i}\alpha)}} \tag{5.3}$$

and taking logarithms in (5.2a)–(5.2b) we obtain

$$N_3^* \phi\left(v_k^{(2)}, \frac{\gamma}{2}\right) + \sum_{j=1}^{r} \phi\left(v_k^{(2)} - v_j^{(1)}, \frac{\gamma}{2}\right) - \sum_{i=1}^{s} \phi(v_k^{(2)} - v_i^{(2)}, \gamma) = 2\pi I_k^{(2)} \qquad 1 \leqslant k \leqslant s \tag{5.4a}$$

$$N_3 \phi\left(v_k^{(1)}, \frac{\gamma}{2}\right) - \sum_{j=1}^{r} \phi\left(v_k^{(1)}, \frac{\gamma}{2}\right) + \sum_{i=1}^{s} \phi\left(v_k^{(2)} - v_i^{(1)}, \frac{\gamma}{2}\right) = 2\pi I_k^{(1)} \qquad 1 \leqslant k \leqslant r \tag{5.4b}$$

where $I_k^{(1)}$ and $I_k^{(2)}$ are half-integers.

In the thermodynamic limit $N \to \infty$, the roots tend to have continuous distributions. Unlike what happens in other cases, we cannot distinguish between the roots coming from the different types of representations, this can be noted by simple inspection of the equations of the ansatz. Due to that, we define two root densities, one for each level,

$$\rho_l(v_j^{(l)}) = \lim_{N_3 \to \infty} \frac{1}{N_3(v_{j+1}^{(l)} - v_j^{(l)})} \qquad l = 1, 2. \tag{5.5}$$

Let it be

$$Z_{N_3}(v) = \frac{1}{2\pi}\left[\phi\left(v, \frac{\gamma}{2}\right) - \frac{1}{N_3}\sum_{j=1}^{r}\phi(v - v_j^{(1)}, \gamma) + \frac{1}{N_3}\sum_{j=1}^{s}\phi\left(v - v_j^{(2)}, \frac{\gamma}{2}\right)\right] \tag{5.6a}$$

$$Z_{N_3^*}(v) = \frac{1}{2\pi}\left[\phi\left(v, \frac{\gamma}{2}\right) - \frac{1}{N_3^*}\sum_{j=1}^{s}\phi(v - v_j^{(2)}, \gamma) + \frac{1}{N_3^*}\sum_{j=1}^{r}\phi\left(v - v_j^{(1)}, \frac{\gamma}{2}\right)\right]. \tag{5.6b}$$

The no-holes hypothesis for the fundamental state establishes

$$I_{k-1}^{(i)} - I_k^{(i)} = 1 \qquad i = 1, 2 \qquad \text{for all } k \tag{5.7}$$

that implies

$$Z_{N_3}(v_k^{(1)}) = \frac{I_k^{(1)}}{N_3} \tag{5.8a}$$

$$Z_{N_3^*}(v_k^{(2)}) = \frac{I_k^{(2)}}{N_3^*}. \tag{5.8b}$$

In the thermodynamic limit and for the fundamental state, the derivative of these functions are

$$\sigma^{(1)}(v) \equiv \frac{\mathrm{d}}{\mathrm{d}v}Z_{N_3}(v) \approx \frac{N}{N_3}\rho_1(v) \tag{5.9a}$$

$$\sigma^{(2)}(v) \equiv \frac{\mathrm{d}}{\mathrm{d}v}Z_{N_3^*}(v) = \frac{N}{N_3^*}\rho_2(v). \tag{5.9b}$$

Using the approximation

$$\lim_{N_3 \to \infty} \frac{1}{N_3} \sum_j f(v_j^{(k)}) \simeq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\lambda f(\lambda) \rho_k(\lambda) \tag{5.10}$$

together with (5.9a)–(5.9b) and (5.6a)–(5.6b), we obtain the system of equations

$$\rho_1(\lambda) = \frac{1}{2\pi} \left[\frac{N_3}{N} \phi'\left(\lambda, \frac{\gamma}{2}\right) - \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \phi'(\lambda - \mu, \gamma) \rho_1(\mu) \, d\mu \right.$$
$$\left. + \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \phi'\left(\lambda - \mu, \frac{\gamma}{2}\right) \rho_2(\mu) \, d\mu \right] \tag{5.11a}$$

$$\rho_2(\lambda) = \frac{1}{2\pi} \left[\frac{N_3^*}{N} \phi'\left(\lambda, \frac{\gamma}{2}\right) - \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \phi'(\lambda - \mu, \gamma) \rho_2(\mu) \, d\mu \right.$$
$$\left. + \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \phi'\left(\lambda - \mu, \frac{\gamma}{2}\right) \rho_1(\mu) \, d\mu \right] \tag{5.11b}$$

that can be solved by doing the Fourier transform,

$$\phi(\lambda, \alpha) = \pi + 2\lambda - i \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} e^{2im\lambda - 2|m|\alpha} \tag{5.12a}$$

$$\rho_j(\lambda) = \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} e^{2im\lambda} \hat{\rho}_j(m). \tag{5.12b}$$

Introducing these expressions in integral equations (5.11a)–(5.11b), we obtain the densities in the Fourier space

$$\hat{\rho}_1(m) = 2\frac{N_3}{N} \frac{\sinh(2\gamma|m|)}{\sinh(3\gamma|m|)} + 2\frac{N_3^*}{N} \frac{\sinh(\gamma|m|)}{\sinh(3\gamma|m|)} \tag{5.13a}$$

$$\hat{\rho}_2(m) = 2\frac{N_3}{N} \frac{\sinh(\gamma|m|)}{\sinh(3\gamma|m|)} + 2\frac{N_3^*}{N} \frac{\sinh(2\gamma|m|)}{\sinh(3\gamma|m|)} \tag{5.13b}$$

when $m \neq 0$, and

$$\hat{\rho}_1(0) = \frac{2(2N_3 + N_3^*)}{3N} \tag{5.14a}$$

$$\hat{\rho}_2(0) = \frac{2(2N_3^* + N_3)}{3N} \tag{5.14b}$$

for $m = 0$. We note that for $N_3^* = 0$ we have again the known result for a homogeneous chain. It is interesting to note the complementarity of the solution for $N_3^* = 0$ and the solution for $N_3 = 0$,

$$\hat{\rho}_1(m)|_{N_3=0} = \hat{\rho}_2(m)|_{N_3^*=0} \tag{5.15a}$$

$$\hat{\rho}_2(m)|_{N_3=0} = \hat{\rho}_1(m)|_{N_3^*=0}. \tag{5.15b}$$

In the case $N_3 = N_3^* = N/2$, that corresponds to our alternating chain, the densities are given by

$$\hat{\rho}_1(m) = \hat{\rho}_2(m) = \frac{\cosh \frac{1}{2} m\gamma}{\cosh \frac{3}{2} m\gamma}. \tag{5.16}$$

The free energy is defined by the expression

$$\lim_{N \to \infty} f(\theta, \gamma) = -\frac{1}{N} \lg \Lambda(\theta). \tag{5.17}$$

Then, taking the dominant term in $\Lambda(\theta)$,

$$\Lambda_+(\theta) = [a(\theta)]^{N_3}[\bar{b}(\theta)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - \theta) \tag{5.18}$$

the energy is given in this limit by

$$f(\theta, \gamma) = -\frac{N_3}{N} \ln(a(\theta)) - \frac{N_3^*}{N} \ln(\bar{b}(\theta)) + \frac{i}{N} \sum_{j=1}^{r} \Phi\left(v_j^{(1)} + i\frac{3}{2}\theta, \frac{\gamma}{2}\right). \tag{5.19}$$

Changing the variable $u = 3\theta/2$ and using equations (5.10), (5.13a) and (5.13b), the free energy can be written in the clearer form

$$f(u, \gamma) = -\frac{N_3}{N} \ln(\sinh(u + \gamma)) + \frac{4}{3}\frac{N_3}{N}u + \frac{2N_3}{N} \sum_{m=1}^{\infty} \frac{e^{-m\gamma}}{m} \sinh(2mu) \frac{\sinh(2m\gamma)}{\sinh(3m\gamma)}$$

$$- \frac{N_3^*}{N} \ln\left(\sinh\left(u + \frac{3}{2}\gamma\right)\right) + \frac{2}{3}\frac{N_3^*}{N}u + \frac{2N_3^*}{N} \sum_{m=1}^{\infty} \frac{e^{-m\gamma}}{m} \sinh(2mu) \frac{\sinh(m\gamma)}{\sinh(3m\gamma)}. \tag{5.20}$$

As we can see, the free energy is the sum of the individual contributions of the sites in each representation. So, for $N_3^* = 0$ ($N_3 = 0$), we again obtain the results of the homogeneous case in the representation $\{3\}$ ($\{3^*\}$).

From the free energy, we can obtain the energy density in the fundamental state,

$$\mathcal{E} = -\frac{df}{d\theta}\bigg|_{\theta=0} = -\frac{3}{2}\frac{df}{du}\bigg|_{u=0}. \tag{5.21}$$

Doing the calculation, the result is

$$\mathcal{E} = -\frac{3}{2}\left\{\frac{N_3}{N}\left[-\coth\gamma + \frac{4}{3} + 4\sum_{m=1}^{\infty} e^{-m\gamma} \frac{\sinh(2m\gamma)}{\sinh(3m\gamma)}\right]\right.$$

$$\left. + \frac{N_3^*}{N}\left[-\coth\left(\frac{3}{2}\gamma\right) + \frac{2}{3} + 4\sum_{m=1}^{\infty} e^{-m\gamma} \frac{\sinh(m\gamma)}{\sinh(3m\gamma)}\right]\right\} \tag{5.22}$$

that is again the sum of the individual contributions of each site representations.

We can apply the results to the alternating case ($N_3 = N_3^* = N/2$); the free energy is

$$f^{(\text{alt})}(u, \gamma) = \frac{1}{2} \ln(\sinh(u + \gamma)) - \frac{1}{2} \ln(\sinh(u + \frac{3}{2}\gamma)) + \sum_{m=1}^{\infty} \frac{e^{-m\gamma}}{m} \sinh(2mu) \frac{\cosh(\frac{1}{2}m\gamma)}{\cosh(\frac{3}{2}m\gamma)} \tag{5.23}$$

and the energy density of the fundamental state

$$\mathcal{E}^{(\text{alt})} = \frac{3}{4}\left(\coth\gamma + \coth\left(\frac{3}{2}\gamma\right)\right) + \frac{3}{2} - 3\sum_{m=1}^{\infty} e^{-m\gamma} \frac{\cosh(\frac{1}{2}m\gamma)}{\cosh\frac{3}{2}m\gamma}. \tag{5.24}$$

The solutions we have given, were obtained by taking hyperbolic functions for the solutions (4.2a)–(4.2d) and (4.7a)–(4.7d) of the YBE. By considering the trigonometric solutions of these equations and following the same steps, we find the BA equations,

$$\left[\frac{\sinh(v_k^{(1)} - i\frac{\gamma}{2})}{\sinh(v_k^{(1)} + i\frac{\gamma}{2})}\right]^{N_3} = -\prod_{j=1}^{r} \frac{\sinh(v_k^{(1)} - v_j^{(1)} - i\gamma)}{\sinh(v_k^{(1)} - v_j^{(1)} + i\gamma)} \prod_{l=1}^{s} \frac{\sinh(v_l^{(2)} - v_k^{(1)} - i\frac{\gamma}{2})}{\sinh(v_l^{(2)} - v_k^{(1)} + i\frac{\gamma}{2})} \tag{5.25a}$$

$$\left[\frac{\sinh\left(v_k^{(2)} + i\frac{\gamma}{2}\right)}{\sinh(v_k^{(2)} - i\frac{\gamma}{2})}\right]^{N_3^*} = -\prod_{j=1}^{r} \frac{\sinh(v_k^{(2)} - v_j^{(1)} - i\gamma)}{\sinh(v_k^{(2)} - v_j^{(1)} + i\gamma)} \prod_{l=1}^{s} \frac{\sinh(v_l^{(2)} - v_k^{(2)} - i\frac{\gamma}{2})}{\sinh(v_l^{(2)} - v_k^{(2)} + i\frac{\gamma}{2})}. \tag{5.25b}$$

In this regime, the roots cover all real numbers $(-\infty, \infty)$. Then, defining an analogous function

$$\Phi(x, \alpha) = i \ln \frac{\sinh(x + i\alpha)}{\sinh(x - i\alpha)} \tag{5.26}$$

we can solve the problem again by using the Fourier transform

$$\Phi(\lambda, \alpha) = \pi + \int_{-\infty}^{+\infty} \frac{dk}{k} \sin(k\lambda) \frac{\sin(k(\frac{\pi}{2} - \alpha))}{\sin(k\frac{\pi}{2})} \tag{5.27a}$$

$$\rho_j(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \, e^{ik\lambda} \hat{\rho}_j(k). \tag{5.27b}$$

The no-holes hypothesis for the ground state gives us the densities

$$\hat{\rho}_1(k) = \frac{N_3}{N} \frac{\sinh(k\gamma)}{\sinh(\frac{3}{2}k\gamma)} + \frac{N_3^*}{N} \frac{\sinh(k\frac{\gamma}{2})}{\sinh(\frac{3}{2}k\gamma)} \tag{5.28a}$$

$$\hat{\rho}_2(k) = \frac{N_3}{N} \frac{\sinh(k\frac{\gamma}{2})}{\sinh(\frac{3}{2}k\gamma)} + \frac{N_3^*}{N} \frac{\sinh(k\gamma)}{\sinh(\frac{3}{2}k\gamma)} \tag{5.28b}$$

and the free energy becomes

$$f(u, \gamma) = \frac{N_3}{N} \left\{ -\ln\sin(u + \gamma) + 2\int_0^\infty \frac{dk}{k} \frac{\sinh(ku)\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\sinh(k\gamma)}{\sinh(k\frac{\pi}{2})\sinh(k\frac{3\gamma}{2})} \right\}$$

$$+ \frac{N_3^*}{N} \left\{ -\ln\sin\left(u + \frac{3}{2}\gamma\right) + 2\int_0^\infty \frac{dk}{k} \frac{\sinh(ku)\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\sinh(k\gamma)}{\sinh(k\frac{\pi}{2})\sinh(k\frac{3\gamma}{2})} \right\}. \tag{5.29}$$

The density of energy of the ground state is

$$\mathcal{E} = -\frac{3}{2} \left\{ \frac{N_3}{N} \left[-\cot\gamma + 2\int_0^\infty dk \frac{\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\sinh(k\gamma)}{\sinh(k\frac{\pi}{2})\sinh(k\frac{3\gamma}{2})} \right] \right.$$

$$\left. \times \frac{N_3^*}{N} \left[-\cot\frac{3}{2}\gamma + 2\int_0^\infty dk \frac{\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\sinh(k\gamma)}{\sinh(k\frac{\pi}{2})\sinh(k\frac{3\gamma}{2})} \right] \right\}. \tag{5.30}$$

We can specify these magnitudes for the alternating case $(N_3 = N_3^* = N/2)$; they are

$$f^{(\text{alt})}(u, \gamma) = -\frac{1}{2}\ln\sin(u + \gamma) - \frac{1}{2}\ln\sin\left(u + \frac{3}{2}\gamma\right)$$

$$+ \int_0^\infty \frac{dk}{k} \frac{\sinh(ku)\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\cosh(\frac{k}{4}\gamma)}{\sinh(k\frac{\pi}{2})\cosh(k\frac{3\gamma}{4})} \tag{5.31}$$

and

$$\mathcal{E}^{\text{alt}} = -\frac{3}{4} \left(\cot\gamma - \cot\left(\frac{3}{2}\gamma\right) \right) - \frac{3}{2}\int_0^\infty dk \frac{\sinh(k(\frac{\pi}{2} - \frac{\gamma}{2}))\cosh(k\frac{\gamma}{4})}{\sinh(k\frac{\pi}{2})\cosh(k\frac{3\gamma}{4})}. \tag{5.32}$$

We can describe other quantum numbers of the eigenvectors of the transfer matrix. Let we define the number operators

$$\hat{Y}_1 = \hat{N}_u - \hat{N}_{\bar{u}} \tag{5.33a}$$

$$\hat{Y}_2 = \hat{N}_d - \hat{N}_{\bar{d}} \tag{5.33b}$$

where

$$\hat{N}_\alpha = \sum_{i=1}^{N} 1_1 \otimes \cdots \otimes 1 \otimes (n_\alpha)_i \otimes 1 \otimes \cdots \otimes 1_N \tag{5.34}$$

and

$$n_\alpha|\beta\rangle = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha. \end{cases} \tag{5.35}$$

The operators $\hat{Y}_1$ and $\hat{Y}_2$ commute with the transfer matrix

$$[\hat{Y}_i, \tau(\theta)] = 0 \qquad i = 1, 2. \tag{5.36}$$

The commutation relations with the B-operators are

$$[\hat{Y}_1, B_i(\theta)] = -B_i(\theta) \tag{5.37a}$$

$$[\hat{Y}_2, B_i(\theta)] = \delta_{2,i} B_i(\theta). \tag{5.37b}$$

Then, if we apply $\hat{Y}_1$ and $\hat{Y}_2$ on the state $\Psi(\mu)$, obtained by the application of r operators B to the pseudovacuum state $\|0\rangle$ in the first step and s operators in the second step, we find

$$\hat{Y}_1 \Psi(\mu) = (N_3 - r)\Psi(\mu) \tag{5.38a}$$

$$\hat{Y}_2 \Psi(\mu) = (r - s)\Psi(\mu) \tag{5.38b}$$

we have the quantum numbers of this problem as

$$N_u - N_{\bar{u}} = N_3 - r \tag{5.39a}$$

$$N_d - N_{\bar{d}} = r - s \tag{5.39b}$$

and obviously

$$N_s - N_{\bar{s}} = s - N_3^* \tag{5.40}$$

being N_q the eigenvalues of $\hat{N}_q (q = u, \bar{u}, d, \bar{d}, s, \bar{s})$.

In the thermodynamic limit the fundamental state is characterized by

$$\left(\frac{r}{N}\right)_{N\to\infty} = \int_{-A}^{A} \rho_1(\lambda)d\lambda = \frac{2N_3 + N_3^*}{3N} \tag{5.41a}$$

$$\left(\frac{s}{N}\right)_{N\to\infty} = \int_{-A}^{A} \rho_2(\lambda)d\lambda = \frac{N_3 + 2N_3^*}{3N} \tag{5.41b}$$

and then

$$\left(\frac{N_u - N_{\bar{u}}}{N}\right)_{N\to\infty} = \frac{N_3 - N_3^*}{3N} \tag{5.42a}$$

$$\left(\frac{N_d - N_{\bar{d}}}{N}\right)_{N\to\infty} = \frac{N_3 - N_3^*}{3N} \tag{5.42b}$$

$$\left(\frac{N_s - N_{\bar{s}}}{N}\right)_{N\to\infty} = \frac{N_3 - N_3^*}{3N}. \tag{5.42c}$$

For $N_3 = 0$ or $N_3^* = 0$ we recuperate the no-mixing chain results.

In the alternating chain $(N_3 = N_3^* = N/2)$ we obtain

$$\left(\frac{N_u - N_{\bar{u}}}{N}\right)_{N\to\infty} = \left(\frac{N_d - N_{\bar{d}}}{N}\right)_{N\to\infty} = \left(\frac{N_s - N_{\bar{s}}}{N}\right)_{N\to\infty} = 0. \tag{5.43}$$

This proves that the fundamental state is formed by pairs $u\bar{u}$ or $d\bar{d}$ or $s\bar{s}$.

These methods can be easily generalized to higher representations of su(3); the only change is in the g functions (that we call source functions): they change according to the highest weight of the representation. The generalization to su(n) is also straightforward, but laborious; the MBA will have $(n-1)$ steps and will be described by $(n-1)$ source functions, related by $(n-1)$ Bethe equations.

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